ELC 4351: Digital Signal Processing

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Sampling Dilemma

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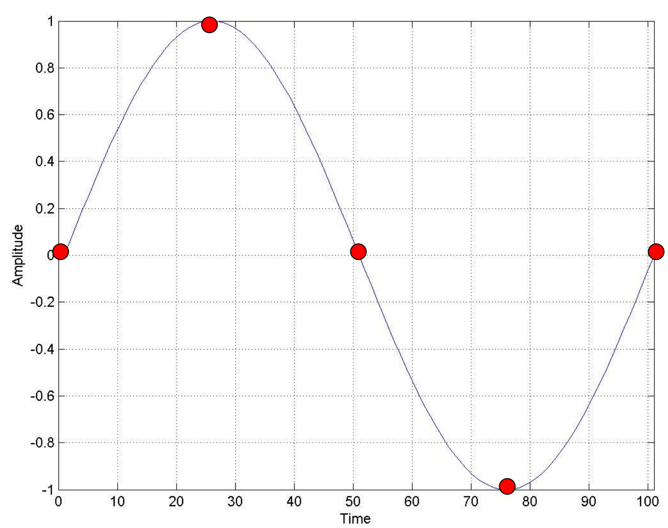
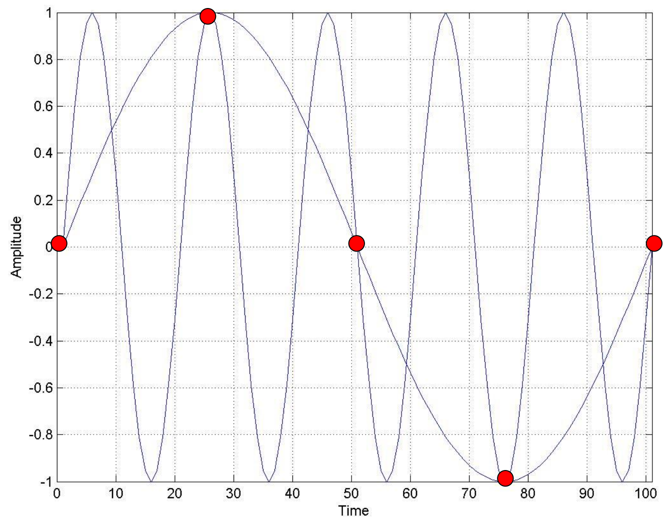


Sampling Dilemma



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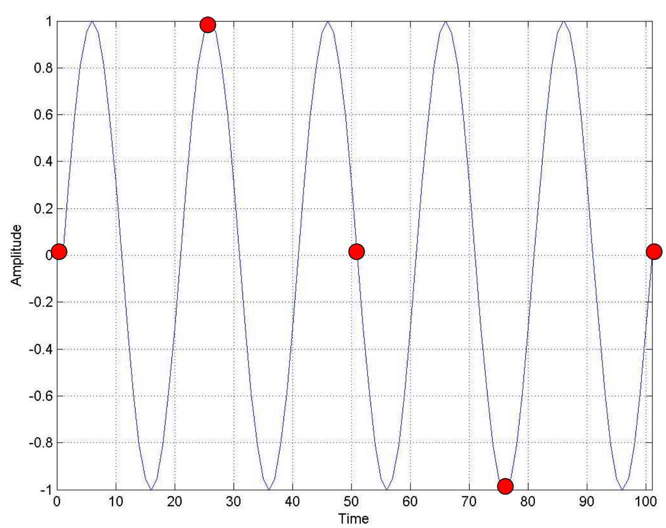
Sampling Dilemma
Sampling Theory
Sampling Theory Proof
Aliasing
Reconstruction

Sampling Dilemma



The Theory

Sampling Theorem

If a signal $x(t)$ contains no frequency components for frequencies above $f = W$ hertz, then it is completely described by instantaneous sample values uniformly spaced in time with period $T_s \leq 1/(2W)$.

That is, the sampling frequency $f_s = 1/T_s$ needs to satisfy

$$\boxed{f_s \geq 2W}$$

The frequency $2W$ is referred to as the *Nyquist frequency*.

ELC 4351: Digital Signal Processing

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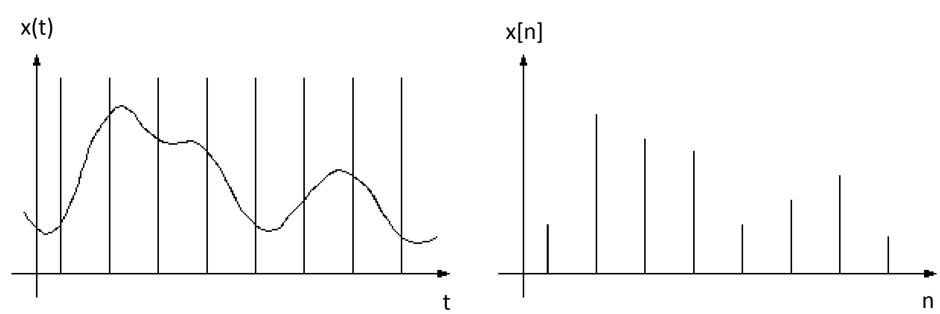
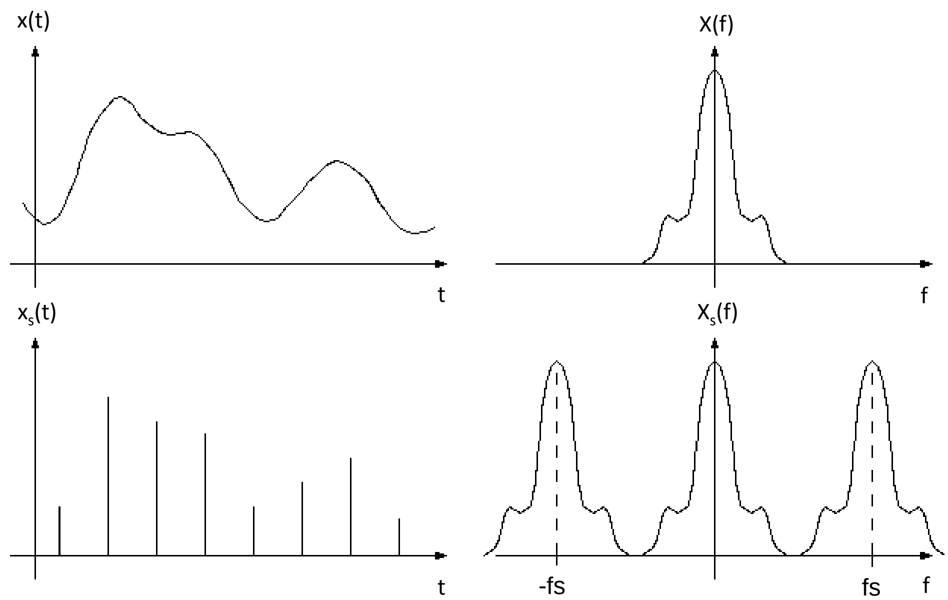
The Proof



Suppose that $x(t)$ is a continuous-time signal. $x[n]$ is the discrete-time signal that consists of samples of $x(t)$ with a sampling period T_s. Therefore,

$$x[n] = x(nT_s), \qquad -\infty < n < \infty.$$



ELC 4351: Digital Signal Processing

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Sampling Theory

Sampling Theory Proof

Aliasing

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- Suppose that the Fourier transform of $x(t)$ is $X(f)$. That is,

$$X(f) = \mathcal{F}\{x(t)\}.$$

- The continuous-time representation of the sampled signal is

$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \end{aligned}$$

where $\delta(t)$ is the Dirac delta function.

- The Fourier transform of $x_s(t)$ is $X_s(f)$, which can be calculated as

$$
\begin{aligned}
X_s(f) = \mathcal{F}\{x_s(t)\} &= X(f) \otimes \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right\} \\
&= X(f) \otimes \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)\right] \\
&= f_s X(f) \otimes \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \\
&= f_s \sum_{n=-\infty}^{\infty} X(f) \otimes \delta(f - nf_s) \\
&= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)
\end{aligned}
$$

where, $f_s = 1/T_s$ is the sampling frequency.

ELC 4351: Digital Signal Processing

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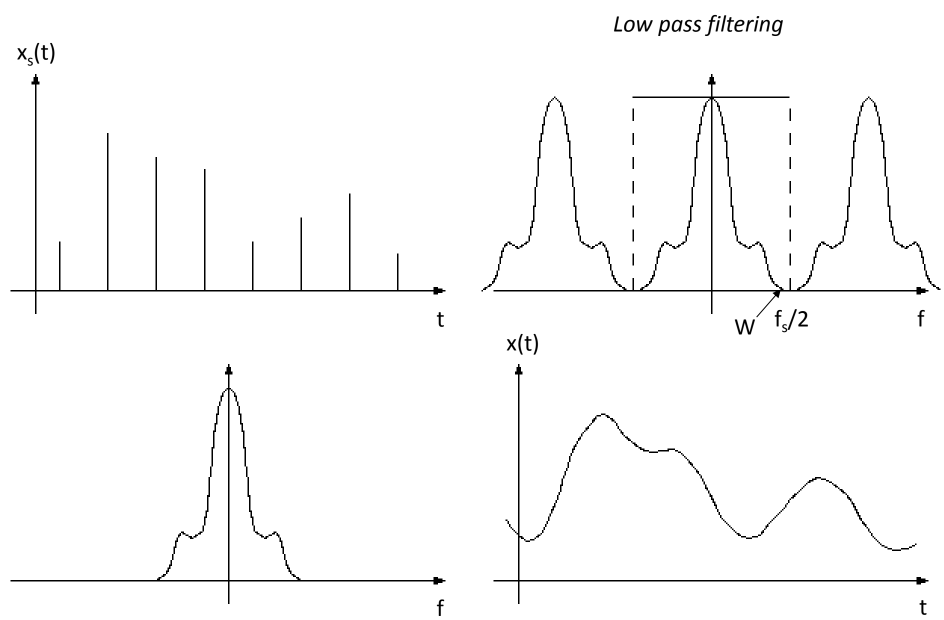

- In order to reconstruct the original signal $x(t)$, we need to pass the sampled signal $x_s(t)$ through an ideal low-pass filter (rectangular function in frequency) to remove the high-frequency replicas.

- A prefect $X(f)$ can be extracted by applying the rectangular function for filtering only when

$$f_s/2 \geq W$$

where W is the largest frequency component in signal $x(t)$. □

ELC 4351: Digital Signal Processing

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Sampling Theory

Sampling Theory Proof

Aliasing

Reconstruction

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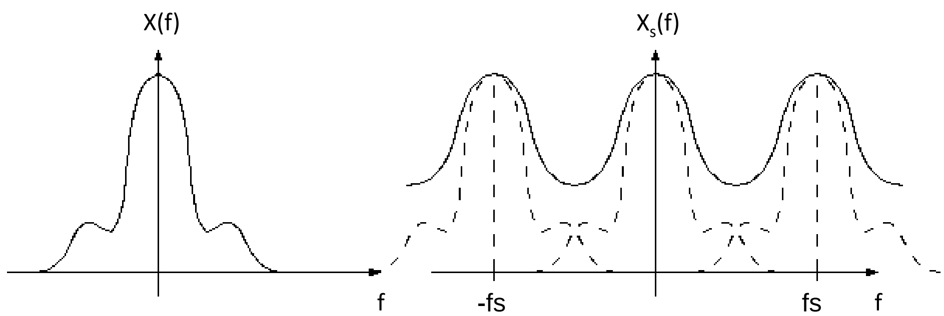

Sampling rate f_s is smaller than the Nyquist rate $2W$.

Reconstruction

$$H(\omega) = \begin{cases} T, & -\frac{\pi}{T} \le \omega \le \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases}$$

$$T = T_s = 1/f_s$$

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} T e^{j\omega t} d\omega \\ &= \frac{T \sin(\pi t/T)}{\pi t} = \text{sinc}\left(\frac{t}{T}\right) \end{aligned}$$

$$\text{sinc}(\tau) = \frac{\sin(\pi\tau)}{\pi\tau}$$

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$$\begin{aligned} x_r(t) &= x_s(t) \otimes h(t) \\ &= \left(\sum_{k=-\infty}^{\infty} x[k]\delta(t - kT) \right) \otimes h(t) \\ &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\delta(\tau - kT) h(t - \tau) d\tau \\ &= \sum_{k=-\infty}^{\infty} x[k] h(t - kT) \\ &= \sum_{k=-\infty}^{\infty} x[k] \,\text{sinc}\left(\frac{t - kT}{T}\right) \end{aligned}$$